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Birkhoff Interpolation of Entire Functions

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We study the interpolation of certain classes of entire functions and their derivatives at the nonnegative integers, which, under certain conditions, reduces to the Birkoff interpolation of polynomials.

1. PÓLYA CONDITIONS

We begin by defining an infinite incidence matrix to be a matrix $E = (e_{j,k})_{0 \leqslant j,k < \infty}$, with $e_{j,k} = 0$ or 1. E defines the interpolation problem

$$f^{(k)}(j) = 0, \qquad \text{if } e_{j,k} = 1, \tag{1}$$

where f is an entire function. If f satisfies (1), we say that *f interpolates E.*

Suppose that Λ is a linear class of entire functions, containing the zero function θ. If the trivial function θ is the only function in Λ which interpolates E, then we say E is *poised* with respect to Λ. Thus, if E is poised with respect to Λ, then the solution $f \in \Lambda$ (if it exists) to the nonhomogeneous equations

$$f^{(k)}(j) = y_j^k, \qquad \text{if } e_{j,k} = 1,$$

for given data $\{y_j^k\}$, is unique in Λ. Therefore, it is of interest to study which arrangements of the entries of 1's will guarantee E to be poised with respect to some class Λ of entire functions. As we shall see, under certain conditions on E and Λ, the interpolation problem (1) will reduce to the Birkhoff interpolation of polynomials.

In order to establish necessary conditions for E to be poised we define

$$m_l = \sum_{j=0}^{\infty} e_{j,l},$$

and

$$M_k = \sum_{l=0}^{k} m_l :$$

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that is, m_ℓ is the number of 1's in column ℓ and M_k is the number of 1's in columns 0 through k. Let ξ be the set of all entire functions and P be the set of all complex polynomials. Similar to the necessary Pólya conditions obtained by Schoenberg [5] in the Birkhoff problem, we have

THEOREM 1. *Let Λ be a linear class of entire functions which satisfies $P \subseteq \Lambda \subseteq \xi$. An incidence matrix E is poised with respect to Λ only if*

$$M_k \geqslant k + 1, \qquad k = 0, 1, \ldots.$$

We refer to the above inequalities as the Pólya conditions. The following argument is similiar to the one given in [5]. In fact, suppose $M_s < s + 1$ for some s. All polynomials of degree $\leqslant s$ satisfy homogeneous Eqs. (1) for each column $k > s$. Since $M_s < s + 1$, columns 0 though s prescribe less than $s + 1$ equations. Hence, there exists a nontrivial polynomial of degree $\leqslant s$ which satisfies homogeneous Eqs. (1) for columns 0 through s and, therefore, interpolates E.

2. Reduction to Polynomial Interpolation

Depending on the arrangement of the entries of 1's in E, the interpolation funtions can be restricted in growth so that E will be poised if and only if E is poised with respect to a set of polynomials. Let P_s be the set of polynomials of degree $\leqslant s$ and let Δ_r be the set of entire functions of exponential type $< r$. If either j or k is negative, then $e_{j,k} = 0$. Let E_n be the truncated matrix $E_n = (e_{j,k})_{j\geqslant 0, 0\leqslant k\leqslant n-1}$. We have the following

LEMMA 1. *Suppose E contains a row, column or diagonal with at most a finite number of* 0's; *that is, let for some nonnegative integers m and n,*

$$0 = e_{m,n-1}, \ 1 = e_{m,n} = e_{m,n+1} = \cdots, \tag{2}$$

or

$$0 = e_{m,n}, \ 1 = e_{m+1,n} = e_{m+2,n} = \cdots, \tag{3}$$

or

$$0 = e_{m-1,n-1}, \ 1 = e_{m,n} = e_{m+1,n+1} = \cdots, \tag{4}$$

respectively. Then E is poised with respect to ξ, Δ_π or Δ_1, respectively, if and only if E (or, equivalently, E_n) is poised with respect to P_{n-1}, where P_{-1} contains only the trivial function θ.

Proof. A polynomial $f \in P_{n-1}$ satisfies all homogeneous Eqs. (1) for

$k \geq n$. Hence it interpolates E if and only if it interpolates E_n. If E is poised with respect to ξ, Δ_π or Δ_1, then E is poised with respect to P_{n-1}, since $P_{n-1} \subset \Delta_1 \subset \Delta_\pi \subset \xi$.

It will be shown that any $f \in \xi$, Δ_π, or Δ_1 which interpolates E, satisfying (2), (3), or (4), respectively, must be a polynomial of degree $\leqslant n-1$, completing the proof of the lemma.

Any entire function f can be written as $f(z) = \sum_{k=0}^{\infty} a_k(z-m)^k$, where $a_k = f^{(k)}(m)/k!$. If E satisfies (2) and f interpolates E, then $f^{(k)}(m) = 0$, for $k = n, n-1, \ldots$ Thus $f(z) = \sum_{k=0}^{n-1} a_k(z-m)^k$ and $f \in P_{n-1}$.

Next, suppose E satisfies (3). By a theorem due to Carlson [cf. 1], if $f \in \Delta_\pi$ and $f(k) = 0$ for $k = 0, 1, \ldots$, then $f = 0$. Let $f \in \Delta_\pi$ be a function which interpolates E. From (3), $f^{(n)}(m+k) = 0$ for $k = 1, 2, \ldots$ Since $F(z) = f^{(n)}(z-m-1)$ is also an element of Δ_π and $F(k) = f^{(n)}(k-m-1) = 0$ for $k = 0, 1, \ldots$, then $F(z) \equiv 0$. Therefore, $f^{(n)}(z) \equiv 0$ and so $f \in P_{n-1}$.

Using Abel series for entire functions one proves (cf. [1, p. 170]) that if $f \in \Delta_1$ and $f^{(n)}(n) = 0$, $n = 0, 1, \ldots$, then $f \equiv 0$. Suppose $f \in \Delta_1$ is a function which interpolates E, and satisfies (4). Then $F(z) = f^{(n)}(z-m)$ is an element of Δ_1 and from (4) we have $F^{(k)}(k) = f^{(n+k)}(m-k) = 0$, $k = 0, 1, \ldots$ Hence $f^{(n)}(z-m) = F(z) \equiv 0$ and $f \in P_{n-1}$, which completes the proof of Lemma 1.

If a polynomial $p \in P_{n-1}$ interpolates E_n, and if the column s of E_n, $0 \leqslant s \leqslant n-1$ has infinitely many ones, then $p^{(s)}(z)$ has infinitely many zeros. This implies that $p^{(s)}(z) \equiv 0$ and so that p is of degree $\leqslant s-1$. This allows us to improve Lemma 1 somewhat. From Lemma 1 and the above remark we deduce

THEOREM 2. *Suppose that E satisfies* (2), (3), *or* (4) *for some integers n, m. Let s be the smallest integer k, $-1 \leqslant k \leqslant n-1$ so that column k of E has infinitely many ones. If there is no such k, we put $s = n$. Then E is poised with respect to ξ, Δ_π, or Δ_1, respectively if and only if E_s is poised with respect to P_{s-1}.*

From the definition of s it follows that the matrix E_s has only finitely many ones. We can omit from E_s all rows that are identically zero. If q is the number of nonzero rows, we obtain in this way a finite $q \times s$ incidence matrix $\hat{E}$. If the number σ of ones in $\hat{E}$ satisfies $\sigma \geqslant s$, our problem reduces to the interpolation of $\hat{E}$ by polynomials from P_{s-1} or, equivalently, from $P_{\sigma-1}$. This is the standard Birkhoff interpolation problem for polynomials, discussed in [3, 4, 6]. However, if $\sigma < s$, Theorem 1 shows that $\hat{E}$ and hence E are not poised with respect to ξ, Δ_π, or Δ_1, respectively.

3. Supplementary Rows

We shall now give sufficient conditions for poisedness which cannot be obtained from theorems about interpolating polynomials. The function $f(z) = \sin(\pi/2m)\, z$ interpolates

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{bmatrix},$$

which contains alternating entries of 0's and 1's in rows 0 and m, even though E contains many 1's and satisfies the Pólya conditions. In view of this example, we shall consider entire functions of smaller growth and infinite incidence matrices which contain supplementary rows defined by

Definition. *Rows $m_1, m_2, \ldots, m_p$ of an infinite incidence matrix E are supplementary rows if*

$$\sum_{k=1}^{p} e_{m_k, j} \geqslant 1, \qquad j = 0, 1, \ldots.$$

In the above example, rows 0 and m are supplementary. We now prove

Theorem 3. *Suppose E contains supplementary rows $m_1 < m_2 < \cdots < m_p$, $p \geqslant 2$, then E is poised with respect to Δ_γ, where $\gamma = c/(e(m_p - m_1))$ and c satisfies $e^c = 2c + 1$ ($c \simeq 1.256$ and $c/e \simeq .462$).*

Proof of Theorem 3. Suppose $f \in \Delta_\gamma$ interpolates E. Let $g(z) = f((m_p - m_1)\, z + m_1)$. Then $g(z)$ is of the same order as f, but has type $\tau < c/e$. In addition,

$$g^{(j)}(z) = (m_p - m_1)^j f^{(j)}((m_p - m_1)\, z + m_1), \qquad \text{for } j = 0, 1, \ldots. \tag{5}$$

We now construct a new matrix H with the same entries as E: however, each row k represents interpolation at $(k - m_1)/(m_p - m_1)$. By (5), g satisfies H. Now, let $d_k = (m_k - m_1)/(m_p - m_1)$ for $k = 1, 2, \ldots, p$. Let row x_k of H

be the row that represents interpolation at d_k. Then, rows $\alpha_1, \ldots, \alpha_p$ of H which correspond to rows $m_1, \ldots, m_p$ of E are supplementary. Note that $d_1 = 0$, $d_p = 1$, $d_k \leqslant 1$ for $1 \leqslant k \leqslant p$ and $g(d_k) = f(m_k)$.

Since rows $\alpha_1, \ldots, \alpha_p$ are supplementary, we can define a function h: $Z^+ \to \{\alpha_1, \ldots, \alpha_p\}$ by the following:

$$h(j) = \alpha_k, \qquad \text{where } k = \max_{1 \leqslant l \leqslant p} \{l : e_{\alpha_l, j} = 1\}.$$

Thus, we have

$$e_{h(j),j} = 1, \qquad j = 1, 2, \ldots.$$

Since g is entire, we can write $g(z) = \sum_{n=0}^{\infty} a_n z^n$. By the convergence of this series, we have for each $k = 1, \ldots, p$,

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (z - d_k + d_k)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{j=0}^{n} \binom{n}{j} (d_k)^{n-j} (z - d_k)^j \\ &= \sum_{j=0}^{\infty} \left(\sum_{n=j}^{\infty} \binom{n}{j} (d_k)^{n-j} a_n \right) (z - d_k)^j \\ &= \sum_{j=0}^{\infty} b_j^{\alpha_k} (z - d_k)^j, \end{aligned}$$

where

$$g^{(j)}(d_k)/j! = b_j^{\alpha_k} = \sum_{n=j}^{\infty} \binom{n}{j} (d_k)^{n-j} a_n, \; k = 1, \ldots, p$$

and $(d_k)^0 = 1$, even if $d_k = 0$, since $g^{(j)}(0)/j! = a_j$.

By the definition of h, we have $b_j^{h(j)} = 0, j = 1, 2, \ldots$. That is, letting $\beta(j) = d_k$, where $h(j) = \alpha_k$, we have

$$\sum_{n=j}^{\infty} \binom{n}{j} (\beta(j))^{n-j} a_n = 0, \qquad j = 1, 2, \ldots. \tag{6}$$

Since g is of growth category $(\rho, \tau) < (1, c/e)$, we have [cf. 1] for large n, $| a_n | \leqslant (\mu/n)^n$, where $0 < \mu < c$. Choose t_0 so that $(\mu/c) < t_0 < 1$ and let $t_1 = (\mu/t_0) < c$. Then, we have for all large n, $| a_n | \leqslant t_0^n \cdot t_1^n/n^n$; hence, $n^n | a_n |/t_1^n \leqslant t_0^n$. Since $t_0 < 1$, the left hand side of the last inequality goes to zero as $n \to \infty$. We may write $a_n = c_n \cdot t_1^n/n^n$, where $\{c_n\}$ is a sequence of complex numbers such that $c_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} | c_n |^2 < \infty$.

Hence, (6) becomes

$$\sum_{n=j}^{\infty} \binom{n}{j} (\beta(j))^{n-j} \cdot \frac{t_1^n}{n^n} \cdot c_n = 0, \qquad j = 1, 2, \ldots, \tag{7}$$

which is an infinite homogeneous system of equations and can be written as the matrix equation $LC = O$, where $C = (c_1, c_2, \ldots,)$ and $L = (\ell_{j,n})$ is given by

$$\begin{aligned} l_{j,n} &= \binom{n}{j} (\beta(j))^{n-j} \cdot t_1^n/n^n, && \text{if } n \geqslant j, \\ &= 0, && \text{if } n < j. \end{aligned}$$

Thus, L is an upper triangular matrix.

It is easy to see that $\sum_{n \geqslant j} |\ell_{j,n}|^2 < \infty$, for each j. By a theorem due to Ching and Chui [cf. 2], if

$$\sum_{n=j+1}^{\infty} l_{j,n} \leqslant l_{j,j}, \qquad j = 1, 2, \ldots.$$

($\ell_{j,n} \geqslant 0, j, n = 1, 2, \ldots,$), then the only solution to (7) is the trivial solution $C = 0$ and, hence, $a_n = 0$, $n = 1, 2, \ldots.$

Thus, we must show

$$\sum_{n=j+1}^{\infty} \binom{n}{j} (\beta(j))^{n-j} \frac{t_1^n}{n^n} \leqslant \frac{t_1^j}{j^j}.$$

Since $\beta(j) \leqslant 1$ for all j,

$$\begin{aligned} \sum_{n=j+1}^{\infty} \binom{n}{j} (\beta(j))^{n-j} \cdot \frac{t_1^n}{n^n} &\leqslant \sum_{n=j+1}^{\infty} \frac{n!}{j!\,(n-j)!} \cdot \frac{t_1^n}{n^n} \\ &\leqslant \sum_{n=j+1}^{\infty} \frac{t_1^n}{(n-j)!\, n^j} \\ &= \frac{t_1^j}{j^j} \cdot \sum_{m=1}^{\infty} \frac{t_1^m}{m!} \left(\frac{j}{m+j}\right)^j. \end{aligned}$$

It can be easily shown that

$$\max_{j=1,\ldots,} \{(j/(m+j))^j\} = 1/(m+1). \tag{8}$$

Using (8), we have

$$\begin{aligned} \sum_{n=j+1}^{\infty} l_{j,n} &\leqslant \frac{t_1^j}{j^j} \sum_{m=1}^{\infty} \frac{t_1^m}{m!} \cdot \frac{1}{m+1} \\ &= \frac{t_1^j}{j^j} \frac{e^{t_1} - 1 - t_1}{t_1}. \end{aligned}$$

Thus, we wish to show that $e^{t_1} - 1 - 2t_1 < 0$. The function $r(x) = e^x - 1 - 2x$ has two roots, $x = 0$ and $x = c$. Also, $r(x) < 0$ for $0 < x < c$ and $r(x) > 0$ everywhere else. Since $0 < t_1 < c$, then $(e^{t_1} - 1 - t_1)/t_1 < 1$. Hence,

$$\sum_{n=j+1}^{\infty} l_{j,n} < \frac{t_1^j}{j^j} < l_{j,j},$$

and so $0 = a_1 = a_2 = \cdots$. Thus $g(z) = a_0$. Since rows $x_1, \ldots, x_p$ are supplementary, there exists at least one k such that $e_{\gamma_k,0} = 1$, which implies $g(d_k) = 0$ and so $g \equiv 0$. Therefore, $f(z) = g((z - m_1)/(m_p - m_1)) \equiv 0$, which completes the proof.

4. Final Remarks

It should be noted that we do not know whether or not the growth restriction of type $\gamma < c/e$ in Theorem 3 is tight, due to difficulty in finding a counterexample.

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